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Lakatos and Hersh on Mathematical Proof*

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Abstract

The concept of Mathematical Proof has been controversial for the past few decades. Different philosophers have offered different theories about the nature of Mathematical Proof, among which theories presented by Lakatos and Hersh have had significant similarities and differences with each other. It seems that a comparison and critical review of these two theories will lead to a better understanding of the concept of mathematical proof and will be a big step towards solving many related problems. Lakatos and Hersh argue that, firstly, "mathematical proof" has two different meanings, formal and informal; and, secondly, informal proofs are affected by human factors, such as individual decisions and collective agreements. I call these two thesis, respectively, "proof dualism" and "humanism". But on the other hand, their theories have significant dissimilarities and are by no means equivalent. Lakatos is committed to linear proof dualism and methodological humanism, while Hersh's theory involves some sort of parallel proof dualism and sociological humanism. According to linear proof dualism, the two main types of proofs are provided in order to achieve a common goal: incarnation of mathematical concepts and methods and truth. However, according to the parallel proof dualism, two main types of proofs are provided in order to achieve two different types of purposes: production of a valid sequence of signs (the goal of the formal proof) and persuasion of the audience (the goal of the informal proof). Hersh's humanism is informative and indicates pluralism; whereas, Lakatos' version of humanism is normative and monistic.

Keywords: *formal proof, informal proof, practical proof, humanism, proof dualism.*

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A formalistic definition of "mathematical proof" which is frequently seen in various related courses and textbooks is something like: A finite sequence of sentences in a formal language, arranged by a certain set of rules (each sentence in the sequence is either an axiom or an assumption or follows from the preceding sentences in the sequence by a rule of inference).

But this definition is neither inclusive nor exclusive. It's not inclusive because mathematicians sometimes use the term "proof" to refer to arguments that do not satisfy the formalistic definition. There are visual proofs, computer-assisted proofs and heuristic proofs. On the other hand, the definition is not even exclusive for some mathematicians and philosophers. For example, intuitionists do not accept the validity of non-constructive proofs even though those proofs can still satisfy the criteria of the formal definition (some lines of the argument can be inferred directly from the principle of excluded middle without being constructed or inferred from preceding formulas in the sequence). As another example, social constructionists believe that an unpersuasive argument should not be called a "proof"; whereas, there is no such condition in the formal definition of proof.

Such and similar problems have motivated theoreticians to look for better or less problematic definitions. One such attempt is to embrace dualism and use the disjunction of the formalistic definition and one or more other complementary definitions to craft a disjunctive definition of "proof". A disjunctive definition of a concept C is the disjunction of a number of subdefinitions, each of which covers only a subset of the concept's extension (whereas, the whole disjunction covers the complete extension). In this case, the disjunctive definition of proof will look like: "A finite sequence of sentences in a formal language, arranged by a certain set of rules OR …". Using disjunctive definitions to define a concept is appropriate when the concept in question is not mononuclear.

The definitions of mathematical proof independently developed by Lakatos and Hersh are both disjunctive definitions. By accepting the formalistic subdefinition as one part of their subjunctive definitions, in contrast to intuitionists, they acknowledge the validity of all classical arguments, including the non-constructive proofs. However, in contrast to formalists and Platonists, by adding some humanistic subdefinition of proof, which takes either psychological or sociological parameters into account, they evidently highlight the role of human factors in any acceptable definition of proof. In their account, the inadequacy of the classical (formalistic) definition was mainly caused by neglecting such human factors. The purpose of this essay is to introduce Lakatos' and Hersh's definitions, to compare them with a critical approach and to show that despite having similarities in regard to their dualist and humanistic approaches, their definitions have significant dissimilarities and are by no means equivalent.

It's worth noting that the controversy over the definition of proof is conceptual. This kind of controversy can be solved (though never fully resolved or settled) by a critical comparison of rival theories and definitions or sometimes even by introducing new definitions. A solution in this case consists of clarifying the philosophical and logical implications of the controversial concept in question, eliminating misunderstandings and getting closer to a mutual understanding between the proponents of the rival theories. In this sense, it seems that a critical study and comparison of these two theories will help us achieve a better understanding of the concept of mathematical proof and mathematics (in general) and even make a big step towards solving some relevant problems (about proof) in Paramathematics[1] and computer sciences.

Hersh's Theory

In "The Mathematical Experience" (1981) Ruben Hersh and Phillip Davis use a fictional dialogue between an ideal mathematician (I.M.) and a student (Student) to examine the concept of proof. Student asks I.M. what a mathematical proof consists of.

> I.M.: […] Anyhow, what you do is, you write down the axioms of your theory in a formal language with a given list of symbols or alphabet. Then you write down the hypothesis of your theorem in the same symbolism. Then you show that you can transform the hypothesis step by step, using the rules of logic, till you get the conclusion. That's a proof.

> Student: Really? That's amazing!… I've never seen that done.
>
> I.M.: Oh! Of course no one ever really does it. It would take forever! You just show that you could do it, that's sufficient (Davis &Hersh, 1990:39).

But Student, who is not convinced with the answer, keeps asking for a definition of proof.

> I.M.: Well, it's an argument that convinces someone who knows the subject.
>
> Student: someone who knows the subject? Then the definition of proof is subjective; it depends on particular persons. …
>
> I.M: No, no. there's nothing subjective about it! Everybody knows what a proof is. Just read some books, take courses from a competent mathematician, and you'll catch on.
>
> …
>
> Student: Then you decide what a proof is, and if I don't learn to decide in the same way, you decide I don't have any aptitude.
>
> I.M: If not me, then who? (Davis &Hersh, 1990:40)

In this dialogue, IM implicitly admits that the formalistic definition of proof does not adequately describe the proofs we use in practice; meanwhile he fails to present an objective definition for it. In return, he offers a criterion (Persuasiveness and institutional legitimacy) to verify the validity of a given proof.

In his "What Is Mathematics, Really?" (1997), Hersh takes a clearer stance:

> Mathematical proof" has two meanings. In practice, it's one thing. In principle, it's another. We show student what proof is in practice. We tell them what it is in principle. […]. Meaning number 1, the practical meaning, is informal, imprecise. Practical mathematical proof is what we do to make each other believe our theorems. It's

> argument that convinces the qualified, skeptical expert. […] meaning number 2, theoretical mathematical proof is formal. […] it's transformation of certain symbol sequences (formal sentences) according to certain rules of logic (modus ponens, etc.). (Hersh, 1997:49)

Olsker adds the following explanation to clarify Hersh's standpoint on the subject:

> The practical meaning implies that proof has a subjective side; the goal of a proof is to convince the mathematical community of the truth of a theorem. That is, mathematics is a human endeavor, since proofs are written, read, understood, verified, and used by humans.(Olsker, 2011:36)

There are three points that need to be taken into consideration: First, we should not think that the informal and imprecise nature of practical proofs makes them non-rigorous as well. Hersh has repeatedly emphasized that the meaning of rigorous proof needs to be refined to include proofs adequately supported with machine computations or numerical evidences as well as those with strong probabilistic algorithms (Hersh, 1997:58).

In his "Proof: Its Nature and Significance" (2008), Detlefsen offers a better understanding of rigor. First of all, he emphasizes that formalization and rigor are mutually independent. "The prevailing view of proof sees rigor as a necessary feature of proof and formalizability as a necessary condition of rigor." (Detlefsen, 2008:16) "Rigorous proof, on this view, is reasoning all of whose inferences track purely logical relations between concepts. (Detlefsen, 2008:17)

This can explain the traditional and common misbelief that rigor and precision of a mathematical proof should necessitate its independence of empirical experiences as well as intuition, natural language and common sense (and consequently, the belief that "rigorous" and "formal" are co-extensional). Detlefsen holds that the traditional belief mentioned above stemmed from the dominance of formalism and syntacticalism during the late 19 and early 20th centuries. He then offers an alternate account of mathematical precision by disconnecting it from formalization and explaining it in terms of explanatory content. He argues that a similar viewpoint exists even in the traditional approach according to which a mathematical argument is considered to be of the highest precision only when it has the highest explanatory potential. In fact, while avoiding the possible gaps in an argument, we achieve the highest level of certainty only when the premises of the argument can explain its results successfully.

In Detlefsen's account, the precision of an argument is tied with its explanatory potential. The more precise and rigorous an argument can be, the better its premises can explain the result. When the explanatory potential becomes more transparent, we are more content to add missing information to close the gaps between premises and the result; while on the other hand, adding more formalization to the argument will decrease the level of transparency and precision along with it. Hence: "A reexamination of the commonly presumed connection(s) between rigor and formalization would thus seem to be in order." (Detlefsen, 2008:19)

It can now be seen that practical proofs, for their high transparency, can be of such a great and unmatchable help for understanding and developing mathematics in its generality as well as specific procedures like hypothesizing, finding contradictions, creative reasoning and conceptualizing. Understanding something is nothing but explaining it in a successful and efficient way. This result leads us to the second important point: We should not think of practical proofs as "pseudo-proofs", "immature proofs", "fake", "second class" or anything of the sort. On the contrary, practical proofs make mathematics progress. They are what mathematicians call "proof". Formal proofs should be actually called "logical proofs" rather than "mathematical proofs": "Real-life proof is informal" (Hersh, 1997:57) Therefore, practical proofs, despite being informal, are rigorous and explanatory. They play an unmatchable role in the progress of mathematics and they are what mathematicians refer to as "proof" in practice.

The third point in understanding Hersh's theory is that proofs in addition to their logical and lingual aspects have mental aspects as well. Hersh uses the terms "convince", "convincing" and "being convinced" to highlight this aspect of the debate. Olsker has also correctly emphasized on the same point. In addition to these all, it also seems that we need to emphasize on the social aspects of proofs. The fourth important point is a proof's institutional legitimacy as distinct from the mental persuasiveness or convincing power. Olsker though seems to have neglected this distinction and mixed the mental and social (and institutional) aspects of the debate together:

> As pointed out by Davis and Hersh above, and by others, when a mathematician reads a proof to determine its validity, he or she makes that determination based on whether or not he or she finds the proof to be convincing. That is, the mathematician makes a judgment based on subjective criteria. The Clay Mathematics Institute, which offers a one million dollar prize for a proof of any one of seven mathematical conjectures, stipulates that any proof must be published and accepted by the community of mathematicians for two years before a prize will be awarded. Because the validity of a proof depends on acceptance by mathematicians, that validity is inherently subjective. (Olsker, 2011:37)

For a proof to be qualified to win a million dollar prize it has to have been published for two years. In a more general sense, a proof can be identifiable and referable among mathematician only if it is published (it won't have objective existence if it's not published). Unlike Olsker (above), it seems to us that the validity of proofs is not "inherently subjective" if it's a matter of social and institutional credibility.

In Hersh's account, not only proofs, but also every mathematical entity is a socially constructed concept. For example, if a singular term in Geometry refers to something objective and abstract for a Platonist and refers to basically nothing for a formalist, for Hersh, it refers to something in "the social - conceptual world" (Davis &Hersh, 1990:19) or "the shared conceptual world" (Davis &Hersh, 1990:163). He adheres to the Kuhnian belief that scientific change is a change in what scientists do in practice, rather than a mere change of theories. Hersh sympathizes with Kitcher in generalizing and extending the idea into mathematics. He emphasizes that Kuhnian approach is powerful and convincing when applied to the history of mathematics. (Hersh, 1997, 225)

In the scope of what we learned so far about Hersh, we can now more easily understand and formulate his disjunctive definition of proof. In Hersh's account, a "proof" should satisfy one of the following two conditions:

- There is a sequence of logically arranged well-formed formulas in a formal system (= formal subdefinition); or

- There is a successfully accomplished practice to convince the community of mathematicians (= practical subdefinition).

In other words, there are two parallel types of proofs:

- Formal proof: A finite sequence of well-formed formulas each of which is either an axiom or an assumption or the product of applying a rule of inference to a preceding formula in the sequence.

- Practical proof: A successful practical attempt or endeavor to convince the community of mathematicians to accept the truth of a claim.

If we label these two subdefinitions with P and Q, the complete definition of proof will be "P or Q" and this is what a disjunctive definition should look like. It's needless to add that Hersh can rely on the second part of his disjunctive definition for justifying or explaining any kind of controversial proofs.

Lakatos' theory

About three decades before Hersh, the Hungarian philosopher, Imre Lakatos made similar claims in his book, "Proofs and refutations: the logic of mathematical discovery" (1957) and his article, "what does mathematical proof prove?" (written between 1959 and 1961). Lakatos uses a fictional dialogue as well, though unlike Hersh who used various historical examples in his imaginary dialogue between S and IM, Lakatos composes a dialogue between a teacher and students which particularly concentrates on the heuristic proofs for two theorems of Euler and Cauchy.

At the beginning, the teacher mentions a conjecture he had discussed with students earlier (Euler's conjecture: For any given polyhedron, if V is the number of vertices, E is the number of edges and F is the number of faces, the equation V-E+F=2 is always true). He says: "We tested it by various methods. But we haven't yet proved it." (Lakatos, 1976:8) Then he presents some kind of heuristic proof for it, which he later calls pre-formal proof.

> Teacher: … I have one [(a proof for this theorem)]. It consists of the following thought-experiment.
>
> *Step 1*: Let us imagine the polyhedron to be hollow, with a surface made of thin rubber. If we cut out one of the faces, we can stretch the remaining surface flat on the blackboard, without tearing it. The faces and edges will be deformed, the edges may become curved, but V and E will not alter, so that if and only if V - E + F = 2 for the original polyhedron, V - E + F = 1 for this flat network … [(fig. 1)].
>
> *Step 2*: Now we triangulate our map — it does indeed look like a geographical map. We draw (possibly curvilinear) diagonals in those (possibly curvilinear) polygons which are not already (possibly curvilinear) triangles. By drawing each diagonal we increase both E and F by one, so that the total V-E+F will not be altered … [(fig. 2)].
>
> *Step 3*: From the triangulated network we now remove the triangles one by one. To remove a triangle we either remove an edge- upon which one face and one edge disappear, or we remove two edges and a vertex - upon which one face, two edges and one vertex disappear… [(fig. 3)].

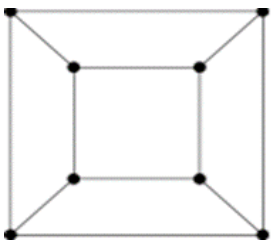

fig. 1

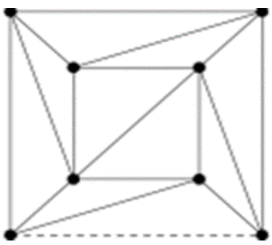

fig. 2

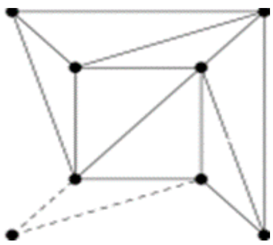

fig. 3

T

> hus if V - E + F = 1 before a triangle is removed, it remains so after the triangle is removed. At the end of this procedure we get a single triangle. For this V - E + F = 1 holds true. Thus we have proved our conjecture.
>
> DELTA: You should now call it a theorem. … .
>
> ALPHA: …, But how am I to know that it can be performed for any polyhedron? For instance, are you sure, Sir, that any polyhedron, *after having a face removed, can be stretched flat on the blackboard*? I am dubious about your first step. (Lakatos, 1976, pp.8-9)

Students Beta and Gamma also shed doubts on second and third steps of the experiment. The teacher then admits that he is not sure of any of those steps and those doubts can be well-grounded.

> TEACHER: I admit that the traditional name 'proof' for this thought-experiment may rightly be considered a bit misleading. I do not think that it establishes the truth of the conjecture.
> DELTA: What does it do then? What do you think a mathematical proof proves?
>
> TEACHER: This is a subtle question which we shall try to answer later. Till then I propose to retain the time-honored technical term 'proof' for a *thought-experiment - or 'quasi-experiment' - which suggests a decomposition of the original conjecture into subconjectures or lemmas*, thus embedded it in a possibly quite distant body of knowledge. Our ‘proof ‘, for instance, has embedded the original conjecture - about crystals, or, say, solids - in the theory of rubber sheets. Descartes or Euler, the fathers of the original conjecture, certainly did not even dream of this.” (Lakatos, 1976:10)

Lakatos repeats this proof in the formerly mentioned article (“what does mathematical proof prove?”) too and this time using a monologue discourse he takes a clear stance in regard to the nature of mathematical proof. In Lakatos account, mathematical proofs are basically of three different types:

- Pre-formal proofs (the first type): These are proofs presented before a formal system is fully developed, just like the proof of the Euler's conjecture we observed above. Lakatos' proof for the Euler's conjecture may look artificial but he shows that this kind of proof is found in abundance in mathematics (another example is the proof for Cauchy's theorem).

- Formal proofs (the second type): These are proofs in a developed formal system. This type of proof fits well in the formalistic definition of proof. An example is the proof of the following equation in Zermelo's formal system: $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

- Post-formal proofs (the third type): These are proofs of meta-theorems of a formal system in absence of any meta-theory or formal meta-system. Examples of this type of proof are the proofs of the undecidability theorems in logic and the principle of duality in projective geometry.

Even though Lakatos says that mathematical proofs are essentially of three different types: pre-formal; formal and the post-formal, it is not hard to see that two of these three types - the pre-formal and post-formal proofs - are both "informal" proofs, as distinct from the formal type of proofs. Lakatos holds that informal proofs render a lower level of certainty compared to formal ones, but they also prove statements and theorems that are clearer and more tangible. They prove things that mathematicians are really interested in. Formal proofs, on the other hand, are absolutely reliable; though sadly, it's not always clear what their reliability is actually about. (Lakatos, 1978:69)

We can summarize Lakatos' theory here. There are two general types of proofs: Formal proofs and informal proofs.

- Formal proof: a process that proves a sentence in a formal system using the system's axioms and rules of inference in the form of a finite sequence of well-formed formulas starting with axioms or premises and ending with the desired result.

- Informal proof: a process that shows the truth of a statement with heuristic reasoning. In other words, it uses a correct mental experience to analyze the main statement to simpler and more evident statements until counter examples (if any) appear, concepts are detected and clarified and eventually an informal mathematical theory is shaped and finally, the desired statement shows up as an outcome of this theory.

Somewhat similar to the points we made about Hersh's theory, there are also three important points about Lakatos' theory:

First and foremost, the same point we made about the rigor and precision of Hersh's practical proofs also hold about Lakatos' informal proofs. If a proof is informal, it doesn't mean that it is imprecise. In particular, if we take Detlefsen's viewpoint about the relation between precision and explanatory power and transparency into account and compare Lakatos' heuristic argument for Euler's theorem with it, we can see that proof is reasonably precise. The cuttings, making triangles and then omitting those triangles one after another can explain the result (V-E+F=2) fairly well. However, in all such heuristic and informal proofs, there is always a chance that some hidden assumptions are neglected or remained unsaid (in this case, the assumption that the polyhedron has no cavity), but this level of fallibility doesn't make it any less proof; it just makes it different from a formal proof and leads to the conclusion that informal proofs (unlike the formal ones) are falsifiable and that's why they can be called "quasi-empirical". Basically, we are facing some synthetic and "a posteriori" element in it, as opposed to something purely analytic and "a priori".

The second important point is that Lakatos has presented historical and empirical evidences for the existence of informal proofs. This is important because Lakatos' proof for Euler theorem might look artificial at the first sight and generalizing the concept to the real world of mathematics might seem dubious. In fact, he uses his example to elaborate the difference between informal and formal proofs, but then he offers real and historical examples as well. In particular, he mentions and explains the heuristic proof for Cauchy's theorem and the concept of uniform convergence. (Lakatos, 1976:144)

The third important point about informal proofs is that in addition to be valid independent from the formal definition of validity, they even play an immensely important role to help mathematicians pick the right formal axioms to construct formal systems. From this point of view, it can be said that informal proofs provide the base and foundation for validity of formal proofs. Lakatos holds that pre-formal proofs are an important part of the procedure to make informal mathematical theories, which are the main base and source for construction of formal systems. (Lakatos, 1978:62)

Similarities and differences

Hersh and Lakatos accept the formalistic definition of proof, though they don't believe it to be adequate. Therefore, each of them adds an informal subdefinition to it and constructs a new definition in form of a disjunction. The idea behind this type of disjunctive definition is rooted in a more basic and fundamental doctrine in regard to the nature of any given concept (such as proof), which can be called "Dualism". Dualism, in general, is any theory that recognizes two and only two independent and mutually irreducible principles or entities or meanings, which are sometimes complementary and sometimes in conflict. I am using the word "proof dualism" with the same considerations: "mathematical proof" has two and only tow independent and mutually irreducible meanings. We can see that Lakatos and Hersh are proof dualist. Had they believed that all proofs share a single and unique essence, they would have definitely formulated a new, but unique definition to replace the classical and traditional one. That's actually what Kitcher does. He says: "Proofs are sequences of sentences which […] codify psychological processes which can produce a priori knowledge of the theorem proved." (Kitcher, 1989:37)

In addition to the structural similarity mentioned above, both Hersh and Lakatos share the idea that mathematics is a human activity. Lakatos emphasizes "mathematical activity is a human activity" (Lakatos, 1976:146). Hersh calls this understanding of mathematics "Humanism" and says: "I use "humanism" to include all philosophies that see mathematics as a human activity, a product, and a characteristic of human culture and society." (Hersh, 1997: xi) He then presents a list of old and modern humanists in the history, which includes Lakatos as well: Aristotle, Locke, Hume, Mill, Peirce, Sellars, Wittgenstein, Popper, Lakatos, Tymoczko and Kitcher. Hersh is a humanist and calls his own approach "Social-historical approach"

(Hersh, 1997: 24) and says: “Mathematics is human. It's part of and fits into human culture.” (Hersh, 1997: 22)

One of the most important implications of such as approach is that our cultural needs and values play an important role in convincing the community of mathematicians and that the study of the concept of proof cannot be fully accomplished without a cultural and historical study of mathematics. In other words, definition of proof requires historical and sociological elements beside logic and syntax.

> There are different versions of proof or rigor, depending on time, place, and other things. The use of computers in proofs is a nontraditional rigor. Empirical evidence, numerical experimentation, probabilistic proof all help us decide what to believe in mathematics. Aristotelian logic isn’t always the only way to decide. (Hersh, 1997: 22)

That’s why (along with Kitcher) Hersh construes activities of mathematicians in a Kuhnian framework. Lakatos, on the other hand, considers the study of mathematicians’ methodological goals and decisions as complementary to logic and syntax.

> But mathematical activity produces mathematics. Mathematics, this product of human activity, ‘alienates itself’ from the human activity which has been producing it. It becomes a living, growing organism, that *acquires a certain autonomy* from the activity which has produced it; it develops its own autonomous laws of growth, its own dialectic. The genuine creative mathematician is just a personification, an incarnation of these laws which can only rely on human action. (Lakatos, 1976:146)

Let us see how this personification and incarnation is to be rendered by mathematicians. We return to the dialogue between Teacher and Students and proceed from the point Student Gamma proposes a counter example in which the polyhedron has cavity and Euler’s conjecture doesn’t hold for it anymore and the theorem must surrender to the counter example. Student Delta, however, is not happy with this methodological decision:

> DELTA: But why accept the counter example? We proved our conjecture, now it is a theorem. I admit that it clashes with this so-called 'counterexample'. One of them has to give way. But why should the theorem give way, when it has been proved? It is the 'criticism' that should retreat. It is fake criticism. This pair of nested cubes is not a polyhedron at all. It is a monster a pathological case, not a counterexample.
>
> GAMMA: Why not? A polyhedron is a solid whose surface consists of polygonal faces. And my counterexample is a solid bounded by polygonal faces.
>
> TEACHER: Let us call this definition Def. 1 (Lakatos, 1976, pp.15-16)

The teacher and students then continue suggesting better definitions for simple polyhedron trying to amend the shortcomings of the previous definitions at every stage. In the end, after examining various counter examples and changing the definition for 6 times, they came to the agreement that Euler's theorem holds for simple convex polyhedron.[2]

On the surface of the dialogue, Lakatos seems to be describing the procedure of defining the concept of polyhedron by the teacher and students, but at a deeper layer, he is explaining the process of making methodological decisions: When mathematicians encounter counter examples, they refine their auxiliary hypotheses to protect the hard core of their research program (as per negative heuristics), but if these measures fail to resolve the anomalies, they use positive heuristics to adjust their concepts and axioms. Lakatos could have followed to Kant's recommendation and stay in the methodological level, but he proceeds to the ontological level and uses a Hegelian dialectical approach to explain the logic of mathematical discovery (maybe his realism and anti-relativist approach is the reason he makes this step to ontology). He claims that it is mathematics (itself) which is incarnated through the mathematicians' dialogues and decision-making procedures. In fact, for Lakatos, the Hegelian and metaphysical concept of heuristics is the base for the Kantian concept of heuristics in methodology: "The Hegelian language, which I use here, would I think, generally be capable of describing the various developments in mathematics." (Lakatos, 1976:145)

For example, when explaining development of mathematics, Lakatos uses the rational evolution (thesis, antithesis and synthesis) and the technical term, "proof-generated concepts", which are those concepts which are created during a heuristic procedure. Examples of such concepts are "simple polyhedron" (in the previously discussed example) and "uniform convergence" (in the proof of Cauchy's theorem).

Lakatos (unlike foundationalists) doesn't reduce the entire concept of proof to logical and lingual elements; however, (unlike Hersh) he explains these informal characteristics in terms of an informal logic rather than cultural and historical values. Therefore, even though Lakatos and Hersh both reject the platonic and foundationalist viewpoints and emphasize on human and mental characteristics of proof (such as mental experiment, decision making and convincing power), Lakatos has a methodological approach, while Hersh's approach is sociological.

The other difference between Lakatos and Hersh is the relation between formal and informal proofs in their theories. For Lakatos, the different types of proofs are in a linear and longitudinal relation with each other. He has pre-formal, formal and post-formal proofs. Pre-formal proofs develop in informal theories and help those theories develop. On the other hand, formal proofs can only be valid in formal systems that have been created on the basis of informal theories, which owe their development to informal proofs. Finally, post-formal proofs can only exist when formal systems are already developed.

Hersh, on the other hand, puts informal proofs on a par with formal ones and calls them practical proofs. The relation is parallel rather than linear.

In other words, for Lakatos, the informal and formal proofs are in a linear relation inside a single research program, namely mathematics. For Hersh, formal and practical proofs are two parallel but distinct research activities practiced by two different institutions: Formal logic and mathematics.

Summary and conclusion

Mathematicians sometimes use the name “proof” for arguments that do not satisfy the formalistic definition of proof. Visional proofs, computer-assisted proofs and heuristic proofs can be mentioned as examples. Lakatos and Hersh are two Philosophers of mathematics who attempted to present alternate definitions for “proof” to solve this problem. Theories presented by these two philosophers have similarities and dissimilarities:

Similarities:

1- Proof dualism: “Proof” has two different meanings, formal and informal (Hersh prefers to say ‘practical’). Formal proofs are those that (more or less) satisfy the classical definition, while informal proofs are heuristic and rigorous arguments that have been successful in convincing their audience in the community of mathematicians and they are valid exactly in this sense and for this achievement. According to both Lakatos and Hersh, the disjunction of these two definitions results in an overall better definition of “proof”.

2- Humanism: Lakatos and Hersh share the opinion that mathematics (in general) and mathematical proof (in particular) are human activities. Mathematics is affected by mathematicians either by the methodological decisions they make or by the cultural values they embrace. Foundationalism in mathematics indicates a non-human point of view. For logicists, mathematics refers to the objective and abstract world of sets and is independent from mathematicians and their decisions or values.

Dissimilarities:

1- Linear Proof dualism vs. Parallel Proof dualism: Lakatos sees informal and formal proofs in a linear relation with each other and speaks of pre-formal and post-formal proofs; whereas, Hersh holds that informal proofs are (Which he calls “practical proof” to distinguish them from formal proofs) are on a par with formal proofs. Linear proof dualism is based on the assumption that mathematics has some standard pattern of evolutionary or historical development – pre formal, formal and post formal stages. Accordingly, the two main types of proofs are provided in a single organism and in order to achieve a common goal: incarnation of Mathematics (i.e., mathematical concepts and methods and truth). However, according to the parallel proof dualism, two main types of proofs are provided in order to achieve two fundamentally different types of purposes: production of a valid sequence of signs and persuasion of the audience.

2- Methodological vs. Sociological Humanism: To explain the concept of proof as a human activity, Lakatos emphasizes on epistemological and methodological purposes and activities of mathematicians; whereas, Hersh concentrates on psychological and sociological attributes. Lakatos' humanism can be explained in the scope of Hegelian dialectic, while Hersh's humanism can be best understood in the framework of Kuhn's scientific revolutions. Besides, Hersh's humanism is informative and indicates pluralism; whereas, Lakatos' version of humanism is normative and monistic.

Each of these two theories has advantages and disadvantages over each other and compared to other rival theories, which are beyond the scope of this essay. The main goal I accomplished in this article was to introduce and compare Lakatos and Hersh's theories and clarify their fundamental similarities and dissimilarities. As Popper has correctly pointed out, two important steps towards the solution of any philosophical problem are: (1) composing the solutions and ideas in form of theories (Popper, 1996:53) and (2) comparing those theories with each other (Popper, 1996:54).

Notes

1. By "Paramathematics" I refer to any interdisciplinary field that is not a branch of mathematics, but related to it. For example, Philosophy of Mathematics, History of Mathematics, Sociology of Mathematics, Mathematics Education, etc.
2. Without cavity and stretchable to a plane, Ibid, 34-36

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